

Hyperbolic complex contact structures on \mathbb{C}^{2n+1}

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Abstract In this paper we construct complex contact structures on \mathbb{C}^{2n+1} for any $n \geq 1$ with the property that every holomorphic Legendrian map $\mathbb{C} \rightarrow \mathbb{C}^{2n+1}$ is constant. In particular, these contact structures are not globally contactomorphic to the standard complex contact structure on \mathbb{C}^{2n+1} .

Keywords complex contact structures, hyperbolicity, Fatou-Bieberbach domains.

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1. Introduction and main results

Let M be a complex manifold of odd dimension $2n + 1 \geq 3$, where $n \in \mathbb{N} = \{1, 2, \dots\}$. A holomorphic vector subbundle $\xi \subset TM$ of complex codimension one in the tangent bundle TM is a *holomorphic contact structure* on M if every point $p \in M$ admits an open neighborhood $U \subset M$ such that $\xi|_U = \ker \alpha$ for a holomorphic 1-form α on U satisfying

$$\alpha \wedge (d\alpha)^n \neq 0.$$

A 1-form α satisfying this nondegeneracy condition is called a *holomorphic contact form*, and (M, ξ) is a *complex contact manifold*. We shall also write (M, α) when $\xi = \ker \alpha$ holds on all of M . The model is the complex Euclidean space $(\mathbb{C}_{x_1, y_1, \dots, x_n, y_n, z}^{2n+1}, \xi_0 = \ker \alpha_0)$ where α_0 is the standard complex contact form

$$(1.1) \quad \alpha_0 = dz + \sum_{j=1}^n x_j dy_j.$$

By Darboux's theorem, every holomorphic contact form equals α_0 in suitably chosen local holomorphic coordinates at any given point (see e.g. Geiges [11, Theorem 2.5.1, p. 67] for the smooth case and [1, Theorem A.2] for the holomorphic one). This standard case has recently been considered by Alarcón, López and the author in [1]. They proved in particular that every open Riemann surface R admits a proper holomorphic embedding $f: R \hookrightarrow (\mathbb{C}^{2n+1}, \alpha_0)$ as a *Legendrian curve*, meaning that $f^*\alpha_0 = 0$ holds on R . In the same paper, the authors asked whether there exists a holomorphic contact form α on \mathbb{C}^3 which is not globally equivalent to the standard form α_0 (cf. [1, Problem 1.5, p. 4]). In this paper we provide such examples in every dimension.

Theorem 1.1. *For every $n \in \mathbb{N}$ there exists a holomorphic contact form α on \mathbb{C}^{2n+1} such that any holomorphic map $f: \mathbb{C} \rightarrow \mathbb{C}^{2n+1}$ satisfying $f^*\alpha = 0$ is constant. In particular, the complex contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ is not contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$.*

Indeed, a contactomorphism sends Legendrian curves to Legendrian curves, and $(\mathbb{C}^{2n+1}, \xi_0)$ admits plenty of embedded Legendrian complex lines $\mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$. Indeed,

given a point $p = (x_0, y_0, z_0) \in \mathbb{C}^3$ and a vector $\nu = (\nu_1, \nu_2, \nu_3) \in \ker \alpha_0|_p$, the quadratic map $f: \mathbb{C} \rightarrow \mathbb{C}^3$ given by

$$f(\zeta) = (x_0 + \nu_1\zeta, y_0 + \nu_2\zeta, z_0 + \nu_3\zeta - \nu_1\nu_2\zeta^2/2)$$

is a holomorphic Legendrian embedding satisfying $f(0) = p$ and $f'(0) = \nu$.

We expect that our construction actually gives many nonequivalent holomorphic contact structures on \mathbb{C}^{2n+1} ; however, at this time we do not know how to distinguish them. Eliashberg showed that on \mathbb{R}^3 there exist countably many isotopy classes of smooth contact structures [8, 9]. His classification is based on the study of *overtwisted disks* in contact 3-manifolds; it is not clear whether a similar invariant could be used in the complex case.

In order to prove Theorem 1.1, we consider the *directed Kobayashi metric* associated to a contact complex manifold (M, ξ) . Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ denote the open unit disk. Given a holomorphic subbundle $\xi \subset TM$, we say that a holomorphic disk $f: \mathbb{D} \rightarrow M$ is *tangential to ξ* or *horizontal* if

$$f'(\zeta) \in \xi|_{f(\zeta)} \quad \text{holds for all } \zeta \in \mathbb{D}.$$

Consider the function $\xi \rightarrow \mathbb{R}_+$ given for any point $p \in M$ and vector $v \in \xi_p$ by

$$|v|_\xi = \inf \left\{ \frac{1}{|\lambda|} : \exists f: \mathbb{D} \rightarrow M \text{ horizontal, } f(0) = p, f'(0) = \lambda v \right\}.$$

When $\xi = TM$, this is the Kobayashi length of the tangent vector $v \in T_p M$, and its integrated version is the Kobayashi metric on M (cf. Kobayashi [14, 15]). The directed version of the Kobayashi metric was studied by Demailly [5] and several other authors, mainly on complex projective manifolds. More general metrics, obtained by integrating a Riemannian metric along horizontal curves in a smooth directed manifold (M, ξ) , have been studied by Gromov [13] under the name *Carnot-Carathéodory metrics*. (See also Bellaïche [2].) For this reason, we propose the name *Carnot-Carathéodory-Kobayashi metric*, or *CCK metric*, for the pseudodistance function $d_\xi: M \times M \rightarrow \mathbb{R}_+$ defined by

$$(1.2) \quad d_\xi(p, q) = \inf_{\gamma} \int_0^1 |\gamma'(t)|_\xi dt, \quad p, q \in M,$$

where the infimum is over all piecewise smooth paths $\gamma: [0, 1] \rightarrow M$ satisfying $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in \xi_{\gamma(t)}$ for all $t \in [0, 1]$. (By Chow's theorem [4], a horizontal path connecting any given pair of points in M exists when the repeated commutators of vector fields tangential to ξ span the tangent space of M at every point. A discussion and proof of Chow's theorem can also be found in Gromov's paper [13, p. 86 and p. 113]. Another source is Sussman [17, 18].)

The directed complex manifold (M, ξ) is said to be (*Kobayashi*) *hyperbolic* if d_ξ given by (1.2) is a distance function on M (i.e., if $d_\xi(p, q) > 0$ holds for all pairs of distinct points $p, q \in M$), and is *complete hyperbolic* if d_ξ is a complete metric on M . Clearly, the directed Kobayashi metric on (M, ξ) dominates the standard Kobayashi metric on M .

Now, Theorem 1.1 is an obvious corollary to the following result.

Theorem 1.2. *For every $n \in \mathbb{N}$ there exists a holomorphic contact form α on \mathbb{C}^{2n+1} such that the complex contact manifold $(\mathbb{C}^{2n+1}, \xi = \ker \alpha)$ is Kobayashi hyperbolic.*

The contact 1-forms that we shall construct in the proof of Theorem 1.2 are of the form

$$\alpha = \Phi^* \alpha_0$$

where α_0 is the standard contact form (1.1) and $\Phi: \mathbb{C}^{2n+1} \hookrightarrow \mathbb{C}^{2n+1}$ is a *Fatou-Bieberbach map*, i.e., an injective holomorphic map from \mathbb{C}^{2n+1} onto a proper subdomain $\Omega = \Phi(\mathbb{C}^{2n+1}) \subsetneq \mathbb{C}^{2n+1}$ such that $(\Omega, \alpha_0|_\Omega)$ is a hyperbolic contact manifold. Let us describe this construction. Let $C_N > 0$ for $N \in \mathbb{N}$ be a sequence diverging to $+\infty$ and

$$(1.3) \quad K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}_{(x,y)}^{2n} \times C_N \overline{\mathbb{D}}_z.$$

Here, $b\mathbb{D}_{(x,y)}^{2n} \subset \mathbb{C}^{2n}$ denotes the boundary of the unit polydisk in the (x, y) -space and $\overline{\mathbb{D}}_z$ is the closed unit disk in the z direction. Thus, K is the union of a sequence of compact cylinders $K_N = 2^{N-1} b\mathbb{D}_{(x,y)}^{2n} \times C_N \overline{\mathbb{D}}_z$ tending to infinity in all directions. Theorem 1.2 follows immediately from the following two results of possible independent interest. In both results, K is the set given by (1.3).

Proposition 1.3. *If $C_N \geq n2^{3N+1}$ holds for all $N \in \mathbb{N}$ then the domain $\Omega_0 = \mathbb{C}^{2n+1} \setminus K$ is α_0 -hyperbolic. (Here, α_0 is the contact form (1.1).)*

Proposition 1.4. *For every choice of constants $C_N > 0$ there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^{2n+1} \setminus K$.*

Indeed, if a domain $\Omega_0 \subset \mathbb{C}^{2n+1}$ is α_0 -hyperbolic then so is any subdomain $\Omega \subset \Omega_0$. Furthermore, a biholomorphic map $\Phi: \mathbb{C}^{2n+1} \rightarrow \Omega$ is an isometry in the directed Kobayashi metric from the contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ with $\alpha = \Phi^* \alpha_0$ onto the contact manifold (Ω, α_0) . Since (Ω, α_0) is hyperbolic by Proposition 1.3, Theorem 1.2 follows.

Proposition 1.3 is proved in Section 2; the proof uses Cauchy estimates and the explicit expression (1.1) for the standard contact form α_0 . The set K given by (1.3) presents obstacles which impose a limitation on the size of holomorphic α_0 -Legendrian disks.

Proposition 1.4 is a special case of Theorem 3.1 which provides a more general result concerning the possibility of avoiding certain unions of cylinders in \mathbb{C}^n by Fatou-Bieberbach domains. Its proof is inspired by a result of Globevnik [12, Theorem 1.1] who constructed Fatou-Bieberbach domains in \mathbb{C}^n whose intersection with a ball $R\mathbb{B}^n$ for a given $R > 0$ is approximately equal to the intersection of the cylinder $\mathbb{D}^{n-1} \times \mathbb{C}$ with the same ball. His result implies that one can avoid any cylinder K_N in the set K (1.3) by a Fatou-Bieberbach domain Ω . We shall improve the construction so that Ω avoids all cylinders K_N at the same time. For this purpose we will use a sequence of holomorphic automorphisms $\theta_k \in \text{Aut}(\mathbb{C}^n)$ such that the sequence of their compositions $\Theta_k = \theta_k \circ \dots \circ \theta_1$ converges on a certain domain Ω and diverges to infinity on the set K ; hence $K \cap \Omega = \emptyset$. We ensure in addition that each θ_k approximates the identity map on the polydisk $k\overline{\mathbb{D}}^n$, and hence the limit $\Theta = \lim_{k \rightarrow \infty} \Theta_k: \Omega \rightarrow \mathbb{C}^{2n+1}$ is a biholomorphic map of Ω onto \mathbb{C}^{2n+1} .

Several interesting questions remain open. One is whether there exists a *complete hyperbolic* complex contact structure on \mathbb{C}^{2n+1} . Another is whether there exist *algebraic* contact forms α on \mathbb{C}^{2n+1} (i.e., with polynomial coefficients) such that $(\mathbb{C}^{2n+1}, \alpha)$ is hyperbolic. (Our construction only furnishes transcendental examples.) If so, what is the minimal degree of such examples, and for which degrees is a generic (or very generic) contact form hyperbolic? In the integrable case, for affine algebraic and projective manifolds, this is the famous *Kobayashi Conjecture*; see Demailly [6], Brotbek [3] and Deng [7] for recent results on this subject.

Perhaps the most ambitious question is to classify complex contact structures on Euclidean spaces up to isotopy, in the spirit of Eliashberg's classification [8, 9] of smooth contact structures on \mathbb{R}^3 .

Holomorphic contact structures on compact complex manifolds $M = M^{2n+1}$ seem much better understood than those on open manifolds; see for example the paper by LeBrun [16] and the references therein. In particular, the space of all holomorphic contact subbundles of TM , if nonempty, is a connected complex manifold [16, p. 422]. Furthermore, if M is simply connected then any two holomorphic contact structures on M are equivalent via some holomorphic automorphism of M [16, Proposition 2.3]. In particular, the only complex contact structure on the projective space \mathbb{CP}^{2n+1} (up to projective linear automorphisms) is the standard one, given in homogeneous coordinates by the 1-form $\theta = \sum_{j=0}^n (z_j dz_{n+j+1} - z_{n+j+1} dz_j)$. This structure is obtained by contracting the holomorphic symplectic form $\omega = \sum_{j=0}^n dz_j \wedge dz_{n+j+1}$ on \mathbb{C}^{2n+2} with the radial vector field $\sum_{k=0}^{2n+1} z_k \frac{\partial}{\partial z_k}$. Its restriction to any affine chart $\mathbb{C}^{2n+1} \subset \mathbb{CP}^{2n+1}$ is equivalent to the standard contact structure given by (1.1). It follows that the projective space \mathbb{CP}^{2n+1} does not carry any hyperbolic complex contact structures.

2. Hyperbolic contact structures on domains in \mathbb{C}^{2n+1}

In this section we prove Proposition 1.3. For simplicity of notation we consider the case $n = 1$; the same proof applies in every dimension.

Thus, let (x, y, z) be complex coordinates on \mathbb{C}^3 and $\alpha_0 = dz + xdy$ be the standard contact form (1.1) on \mathbb{C}^3 . Recall that $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\overline{\mathbb{D}} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. The definition of the directed Kobayashi metric shows that Proposition 1.3 is an immediate corollary to the following lemma.

Lemma 2.1. *Assume that $C_N \geq 2^{3N+1}$ for every $N \in \mathbb{N}$ and let*

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}_{(x,y)}^2 \times C_N \overline{\mathbb{D}}_z.$$

For every holomorphic α_0 -horizontal disk $f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3 \setminus K$ ($\zeta \in \mathbb{D}$) with $f(0) \in 2^{N_0}\mathbb{D}^3$ for some $N_0 \in \mathbb{N}$ we have the estimates

$$(2.1) \quad |x'(0)| < 2^{N_0+1}, \quad |y'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1}.$$

Proof. Replacing f by the disk $\zeta \mapsto f(r\zeta)$ for some $r < 1$ close to 1 we may assume that f is holomorphic on $\overline{\mathbb{D}}$. Pick a number $N \in \mathbb{N}$ with $N > N_0$ such that $|x(\zeta)| < 2^N$ and $|y(\zeta)| < 2^N$ for all $\zeta \in \overline{\mathbb{D}}$. By the Cauchy estimates applied with $\delta = 2^{-N}$ we then have

$$|y'(\zeta)| < 2^{2N} \quad \text{and} \quad |x(\zeta)y'(\zeta)| < 2^{3N} \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

Since f is a horizontal disk, we have $z'(\zeta) = -x(\zeta)y'(\zeta)$ for $\zeta \in \mathbb{D}$ and hence

$$|z(\zeta)| \leq |z(0)| + \left| \int_0^\zeta xdy \right| < 2^{N_0} + 2^{3N} < 2^{3N+1} \leq C_N \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

From this estimate, the definition of the set K and the fact that $f(\mathbb{D}) \cap K = \emptyset$ it follows that

$$(x(\zeta), y(\zeta)) \notin 2^{N-1}b\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

Since $2^{N-1}b\mathbb{D}^2$ disconnects the disk $2^N\mathbb{D}^2$ and we have $(x(0), y(0)) \in 2^{N_0}\mathbb{D}^2 \subset 2^{N-1}\mathbb{D}^2$, we conclude that

$$(x(\zeta), y(\zeta)) \in 2^{N-1}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$

If $N - 1 > N_0$, we can repeat the same argument with the restricted horizontal disk $f: (1 - 2^{-N})\overline{\mathbb{D}} \rightarrow \mathbb{C}^3$ to obtain

$$(x(\zeta), y(\zeta)) \in 2^{N-2}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - 2^{-(N-1)}.$$

After finitely steps of the same kind we get that

$$(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - \dots - 2^{-(N_0+1)}.$$

Since $2^{-N} + \dots + 2^{-(N_0+1)} < 1/2$, we see that $(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2$ for $|\zeta| \leq 1/2$. Applying once again the Cauchy estimates gives $|x'(0)|, |y'(0)| \leq 2^{N_0+1}$ and hence $|z'(0)| = |x(0)y'(0)| \leq 2^{2N_0+1}$; these are precisely the conditions in (2.1). \square

3. Fatou-Bieberbach domains avoiding a union of cylinders

In this section we prove the following result on avoiding certain closed cylindrical sets in \mathbb{C}^n by Fatou-Bieberbach domains. This includes Proposition 1.4 as a special case.

Theorem 3.1. *Let $0 < a_1 < b_1 < a_2 < b_2 < \dots$ and $c_i > 0$ be sequences of real numbers such that $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} b_i = +\infty$. Let $n > 1$ be an integer and*

$$(3.1) \quad K = \bigcup_{i=1}^{\infty} \left(b_i \overline{\mathbb{D}}^{n-1} \setminus a_i \mathbb{D}^{n-1} \right) \times c_i \overline{\mathbb{D}} \subset \mathbb{C}^n.$$

Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n \setminus K$.

As said in the Introduction, the proof is inspired by [12, proof of Theorem 1.2] to a certain point and is based on the so called push-out method. Since the set K (3.1) is noncompact, the construction of automorphisms used in the proof is somewhat more involved in our case. On the other hand, since our goal is merely to avoid K by a Fatou-Bieberbach domain, and not to approximate a given cylinder as Globevnik did in [12], the construction is less precise in certain other aspects.

Proof. We denote by $\text{Aut}(\mathbb{C}^n)$ the group of all holomorphic automorphisms of \mathbb{C}^n . We first give the proof for $n = 2$ and explain in the end how to treat the general case.

Let (z_1, z_2) be complex coordinates on \mathbb{C}^2 , and let $K = K_1$ be the set (3.1). Up to a dilation of coordinates, we may assume without loss of generality that $a_1 > 1$.

Pick sequence $\epsilon_k \in (0, 1)$ satisfying $\sum_{k=1}^{\infty} \epsilon_k < +\infty$. We shall construct sequences of automorphisms $\phi_k, \psi_k \in \text{Aut}(\mathbb{C}^2)$ ($k \in \mathbb{N}$) of the following form:

$$(3.2) \quad \phi_k(z_1, z_2) = (z_1, z_2 + f_k(z_1)), \quad \psi_k(z_1, z_2) = (z_1 + g_k(z_2), z_2),$$

where f_k and g_k are suitably chosen entire functions on \mathbb{C} to be specified. Set

$$(3.3) \quad \theta_k = \psi_k \circ \phi_k, \quad \Theta_k = \theta_k \circ \dots \circ \theta_1, \quad k \in \mathbb{N}.$$

We will also ensure that for every $k \in \mathbb{N}$ we have

$$|\theta_k(z) - z| < \epsilon_k \quad \text{for } z \in k\overline{\mathbb{D}}^2.$$

Granted the last condition, it follows (cf. [10, Proposition 4.4.1 and Corollary 4.4.2]) that the sequence $\Theta_k \in \text{Aut}(\mathbb{C}^2)$ converges uniformly on compacts in the open set

$$\Omega = \bigcup_{k=1}^{\infty} \Theta_k^{-1}(k\mathbb{D}^2) = \{z \in \mathbb{C}^2 : (\Theta_k(z))_{k \in \mathbb{N}} \text{ is a bounded sequence}\}$$

to a biholomorphic map $\Theta = \lim_{k \rightarrow \infty} \Theta_k : \Omega \rightarrow \mathbb{C}^2$ of Ω onto \mathbb{C}^2 . We will also ensure that

$$(3.4) \quad |\Theta_k(z)| \rightarrow +\infty \quad \text{for all points } z \in K,$$

and hence $K \cap \Omega = \emptyset$. This will prove the theorem when $n = 2$.

We begin by explaining how to choose the first two maps ϕ_1 and ψ_1 ; all subsequent steps will be analogous. Set $b_0 = 1$. Pick a sequence r_j satisfying $b_{j-1} < r_j < a_j$ for all $j = 1, 2, \dots$. Let $N_j \in \mathbb{N}$ be a sequence of integers to be specified later. Set

$$f(\zeta) = \sum_{j=1}^{\infty} \left(\frac{\zeta}{r_j} \right)^{N_j}.$$

This function will define the first automorphism ϕ_1 (cf. (3.2)). Let $f_i(\zeta) = \sum_{j=1}^i \left(\frac{\zeta}{r_j} \right)^{N_j}$ denote the i -th partial sum of the series defining $f(\zeta)$, where we set $f_0 = 0$. By choosing the exponent N_i big enough, we can ensure that the summand $(\zeta/r_i)^{N_i}$ is arbitrarily small on the disk $b_{i-1}\overline{\mathbb{D}}$ and is arbitrarily big on the annulus

$$(3.5) \quad A_i := b_i\overline{\mathbb{D}} \setminus a_i\mathbb{D} = \{\zeta : a_i \leq |\zeta| \leq b_i\}.$$

In particular, we may ensure that for every $i \in \mathbb{N}$ we have

$$(3.6) \quad \sup_{|\zeta| \leq b_{i-1}} \left| \frac{\zeta}{r_i} \right|^{N_i} < 2^{-i-1}\epsilon_1.$$

It follows that the power series defining $f(\zeta)$ converges on all of \mathbb{C} and satisfies

$$(3.7) \quad \sup_{|\zeta| \leq b_{i-1}} |f(\zeta) - f_{i-1}(\zeta)| < 2^{-i}\epsilon_1, \quad i \in \mathbb{N}.$$

Note that the inequalities (3.6) and (3.7) persist if we increase the exponents N_i . We can inductively choose the sequence $N_i \in \mathbb{N}$ to grow fast enough such that the following inequalities hold for every $i \in \mathbb{N}$ with an increasing sequence of numbers $M_i \geq i + 1$:

$$(3.8) \quad \sup_{|\zeta| \leq b_{i-1}} |f_{i-1}(\zeta)| + c_{i-1} + \epsilon_1 < M_i < \inf_{\zeta \in A_i} \left(\left| \frac{\zeta}{r_i} \right|^{N_i} - |f_{i-1}(\zeta)| \right) - c_i - \epsilon_1.$$

(Recall that A_i is the annulus (3.5). Here, $c_0 \geq 0$ is arbitrary while $c_i > 0$ for $i \in \mathbb{N}$ are the constants in the definition (3.1) of the set K .) In view of the inequalities (3.6), (3.7) and (3.8) there exist numbers $\beta_{i-1} < \alpha_i$ such that for all $i \in \mathbb{N}$ we have

$$(3.9) \quad \sup_{|\zeta| \leq b_{i-1}} |f(\zeta)| + c_{i-1} < \beta_{i-1} < M_i < \alpha_i < \inf_{\zeta \in A_i} |f(\zeta)| - c_i.$$

This gives increasing sequences $0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$ diverging to ∞ . Set

$$\phi_1(z_1, z_2) = (z_1, z_2 + f(z_1)).$$

The right hand side of (3.9) shows that for every point $z = (z_1, z_2) \in A_i \times c_i\overline{\mathbb{D}}$ we have

$$|z_2 + f(z_1)| \geq |f(z_1)| - c_i > \alpha_i,$$

while the left hand side of (3.9) gives

$$|z_2 + f(z_1)| \leq c_i + |f(z_1)| < \beta_i.$$

Since these inequalities hold for every $i \in \mathbb{N}$, it follows that

$$\phi_1(K) \subset L := \bigcup_{i=1}^{\infty} b_i \overline{\mathbb{D}} \times (\beta_i \overline{\mathbb{D}} \setminus \alpha_i \mathbb{D}) \subset \mathbb{C}^2.$$

Note that the set L is of the same kind as K (3.1) with the reversed roles of the variables, i.e., the cylinders in L are horizontal instead of vertical. Furthermore, since the sequence α_i is increasing and $\alpha_1 > M_1 \geq 2$ by (3.9), we also see that

$$L \cap (\mathbb{C} \times 2\overline{\mathbb{D}}) = \emptyset.$$

The same argument as above with the set L furnishes a shear automorphism

$$\psi_1(z_1, z_2) = (z_1 + g(z_2), z_2)$$

for some $g \in \mathcal{O}(\mathbb{C})$ (cf. (3.2)) and a set K_2 of the same kind as $K = K_1$ (3.1) (this time again with vertical cylinders) such that, setting $\theta_1 := \psi_1 \circ \phi_1 \in \text{Aut}(\mathbb{C}^2)$, we have

$$(3.10) \quad \theta_1(K_1) \subset K_2, \quad K_2 \cap 2\overline{\mathbb{D}}^2 = \emptyset, \quad \sup_{z \in \overline{\mathbb{D}}^2} |\theta_1(z) - z| < \epsilon_1.$$

Continuing inductively, we find a sequence of automorphisms $\theta_k \in \text{Aut}(\mathbb{C}^2)$ and of closed sets $K_k \subset \mathbb{C}^2$ of the form (3.1) such that for every $k \in \mathbb{N}$ we have

$$(3.11) \quad \theta_k(K_k) \subset K_{k+1}, \quad K_k \cap k\overline{\mathbb{D}}^2 = \emptyset, \quad \sup_{z \in k\overline{\mathbb{D}}^2} |\theta_k(z) - z| < \epsilon_k.$$

Each step of the recursion is of exactly the same kind as the initial one. This implies that

$$\Theta_k(K) \subset K_{k+1} \subset \mathbb{C}^2 \setminus (k+1)\overline{\mathbb{D}}^2, \quad k \in \mathbb{N}$$

and hence (3.4) also holds. This completes the proof when $n = 2$.

Suppose now that $n > 2$. In this case, each automorphism $\theta_k = \psi_k \circ \phi_k \in \text{Aut}(\mathbb{C}^n)$ in the sequence (3.3) is a composition of two shear-like maps of the form

$$\begin{aligned} \phi_k(z_1, z_2, \dots, z_n) &= (z_1, z_2 + f_k(z_1), z_3 + f_k(z_2), \dots, z_n + f_k(z_{n-1})), \\ \psi_k(z_1, z_2, \dots, z_n) &= (z_1 + g_k(z_2), z_2 + g_k(z_3), \dots, z_{n-1} + g_k(z_n), z_n). \end{aligned}$$

A suitable choice of entire functions $f_k, g_k \in \mathcal{O}(\mathbb{C})$ ensures as before that condition (3.11) holds for each k (with $\overline{\mathbb{D}}^2$ replaced by $\overline{\mathbb{D}}^n$). We leave the details to an interested reader. Further details in the case $n > 2$ are also available in [12, proof of Theorem 1.2]. \square

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